Determination of measurement uncertainty

according to GUM

Chapter 3:

Main steps to determinate measurement uncertainty according to GUM

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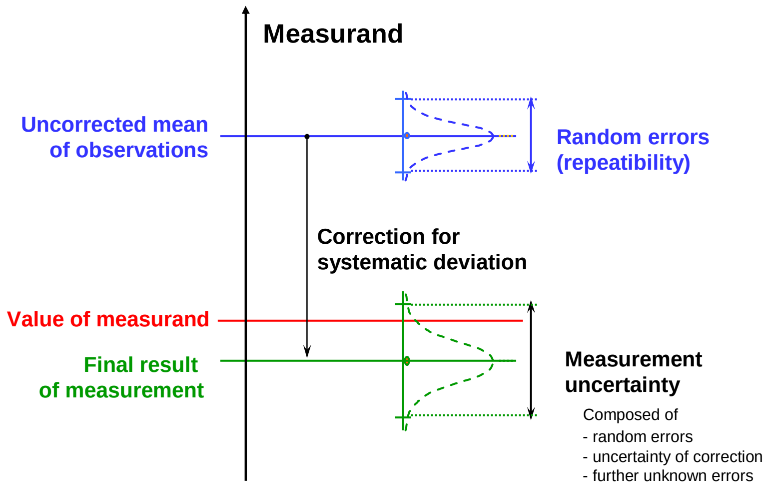
Content

- **Introduction:**
  Principle of measurement uncertainty evaluation
- **The 7 steps to evaluate the measurement uncertainty**
- **Application to an example:**
  Calibration of a volume
- **Summary**

C 3 - 3

Measurement deviations and uncertainty



Source: CENAM

C 3 - 4

Stages of uncertainty evaluation

JCGM 104 “An introduction to the GUM”

- **Formulation stage:**
  - defining the output quantity (measurand) Y
  - identifying the input quantities X_i
  - developing a measurement model $Y = f(X_i)$
  - assigning probability distributions to each X_i

- **Calculation stage:** Obtaining the …
  - expectation of the measurand y
  - standard deviation of the measurand $u(y)$
  - coverage interval for a specific coverage probability $[y_- ; y_+]$

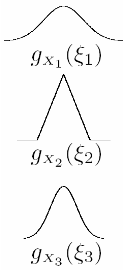
Methods:
a) GUM (approximation)
b) Analytic methods
c) Monte Carlo method

Principle of uncertainty evaluation

Input quantities and their dispersion (uncertainty)

Measurement Model

Measurand and its uncertainty



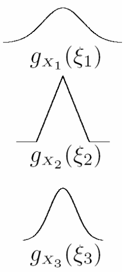
Y=f(X$_i$)

Propagación

y	Expectation
u(y)	Combined standard uncertainty
[y$_-$; y$_+$]	Coverage interval

Principle of GUM

GUM uncertainty framework
for obtaining the **standard uncertainty** of the measurand



x_1, $u(x_1)$

x_2, $u(x_2)$

x_3, $u(x_3)$

Y=f(X$_i$)

Propagation of uncertainties

y

u(y)

[y$_-$; y$_+$]

$$u(x) = \sqrt{E\left((x-\bar{x})^2\right)}$$

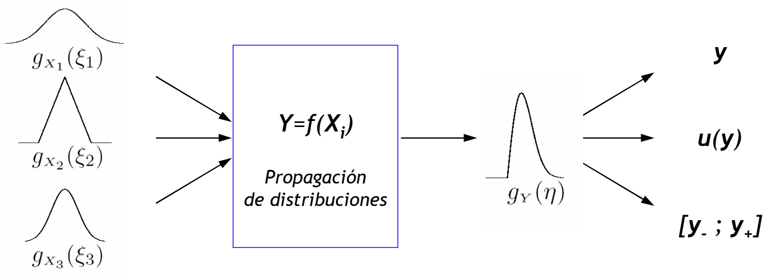
$$u(y) = \sqrt{\sum_i \left(c_i \cdot u(x_i)\right)^2}$$

u standard uncertainty
c_i sensitivity coefficients

Principle:
Propagation of uncertainties

Principle of GUM-S1 (Monte Carlo)

Monte Carlo method of the GUM supplement 1
for obtaining the **probability distribution** of the measurand



Principle:
Propagation of distributions

Advantages of the GUM

➢ The GUM framework presents a standardised and internationally recognised procedure for evaluation and expression of measurement uncertainty**:**
- guidance for users, readily implemented, easily understood
- universal: applicable to all kinds of measurements
- harmonisation facilitates to compare measurement uncertainties obtained in different laboratories

➢ GUM give rules to combine uncertainty contributions evaluated by statistical methods and those evaluated by other means to a single parameter (interval which is supposed to maintain the with a given level of confidence)

➢ GUM is applicable in the majority of measurement situations

➢ GUM permits alternative methods (GUM-G:1.5)
- Analytical methods
- Monte Carlo

➢ GUM enables a simple review and update of the once established uncertainty budget (in difference to Monte Carlo)

Limitations of the GUM

- Distribution of the measurand is not gaussian
- Non-linear model of the measurand and large uncertainties of the input quantities
- Dominant uncertainty contribution form an input quantity with not gaussian distribution
- Asymmetric distribution of an input quantity
- Value and uncertainty of the measurand are of the same order of magnitude

Obtaining the measurement result in 7 steps

Key tasks:

1 **General description** of the measurement
Principle, method and procedure / relevant quantities

2 **Model** for the evaluation
X_1 ... X_N
⇨ $Y=f(X_1,..,X_N)$

3 **Information** about **input quantities**
Methods:
Type - A
Type - B
⇨ x_i , δx_i

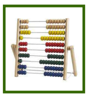
THINK

4 Evaluate **standard uncertainties** of input quantities $u(x_i)$

5 **Combination** of values and uncertainties ⇨ y, $u(y)$

6 Computation of **expanded uncertainty** ⇨ U, k_p, (PDF ?)

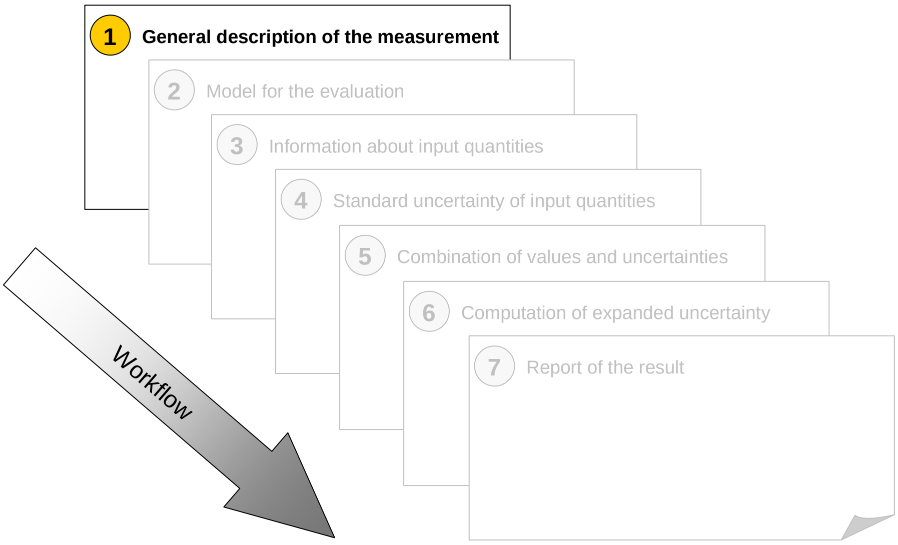
FIXED RULES



7 Report the **result** ⇨ $y \pm U$, k_p, PDF
Plausible? Most relevant contributions? Improvements possible?

THINK & INTERPRET

Following an idea of Bernd R.L. Siebert



General description of the measurement (1)

1) Measurement task and measurand
2) Principle of measurement
3) Method of measurement
4) Measurement procedure

Result of step 1:

- Clear identification of the measurand
- Clear knowledge of the measurement procedure

General description of the measurement

1

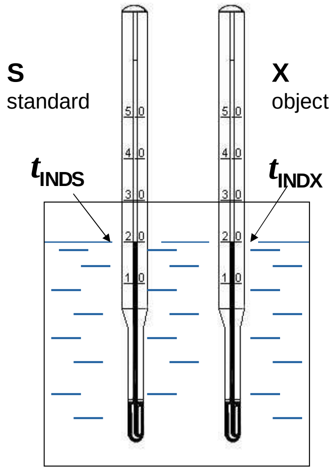
Example: Calibration of a thermometer

1 Measurement task (measurand):
Determination of the deviation of a thermometer at 20°C.

2 Principle of measurement:
Measurement of temperature in a medium at known temperature.

3 Method of measurement:
Comparison of two temperatures (Object {X} und Standard {S}).

4 Measurement procedure:
Comparison in a water bath
Immersing.



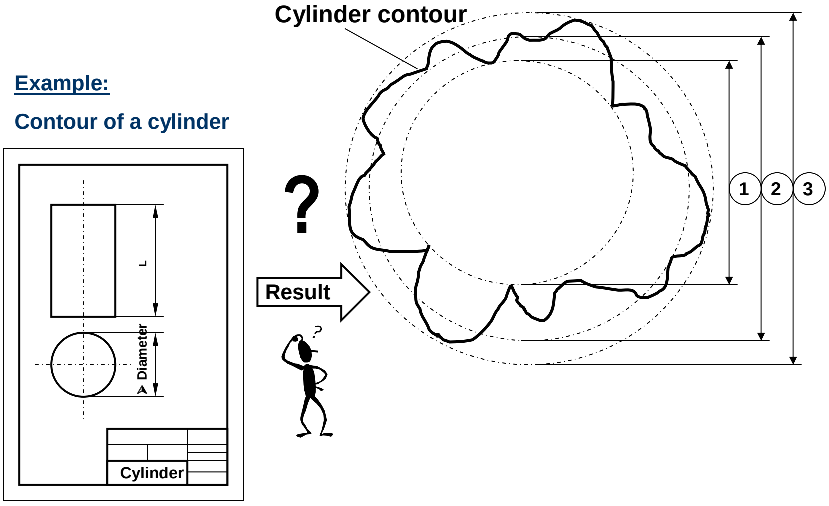
Source: Bernd R.L. Siebert

Definition of the measurand

1

Example:

Contour of a cylinder



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C 3 - 15

Example to be developed:

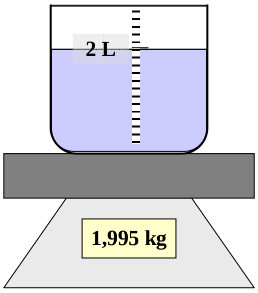
Calibration of a volume

1

1 Measurement task (measurand):
Determination of the volume *V* of a graduated flask of a nominal value of 2 L at a temperature of 20℃.

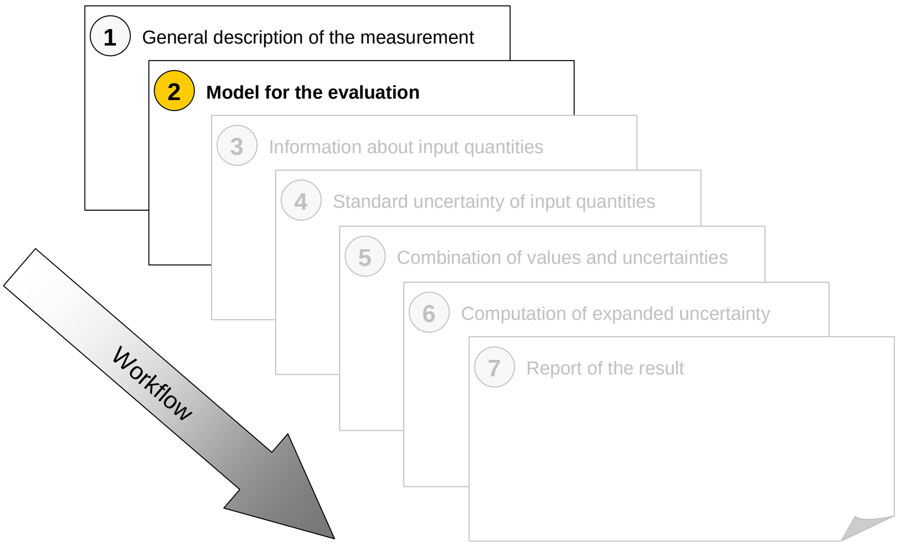
2 Principle of measurement:
Volume = Mass / Density

3 Method of measurement:
Gravimetric calibration



4 Measurement procedure:
Repeated filling of the flask with bi-distilled water weighing the mass *m* of the contained water

C 3 - 16



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Model for the evaluation

2

1) Think about which quantities influence the measurement
2) Structure list of input quantities
3) Develop a mathematical equation relating input quantities to the measurand

Result of step 2:

- Mathematical model of the measurand: $Y = f(X_1, X_2, \ldots X_N)$

Sources of uncertainty in measurements

2

1)	Measurement standard	- calibration - drift since last calibration - finite resolution
2)	Instrument under test	- drift of the instrument parameters - finite resolution
3)	Environmental conditions	- unstable parameters ($t_{ambient}$, h_{rel} , p_{air} …) - imperfect measurement of the parameters - incomplete knowledge of their influence - vibrations, electromagnetic noise, …
4)	Skills of the metrologist	- bias in reading analogue instruments - alignment of the instrument - cleanness
5)	The measurement itself	- Variations in repeated measurements
6)	Definition of the measurand	- incomplete definition and approximations - imperfect realisation

… and many others more

These sources are not necessarily independent, and some of them may contribute to others

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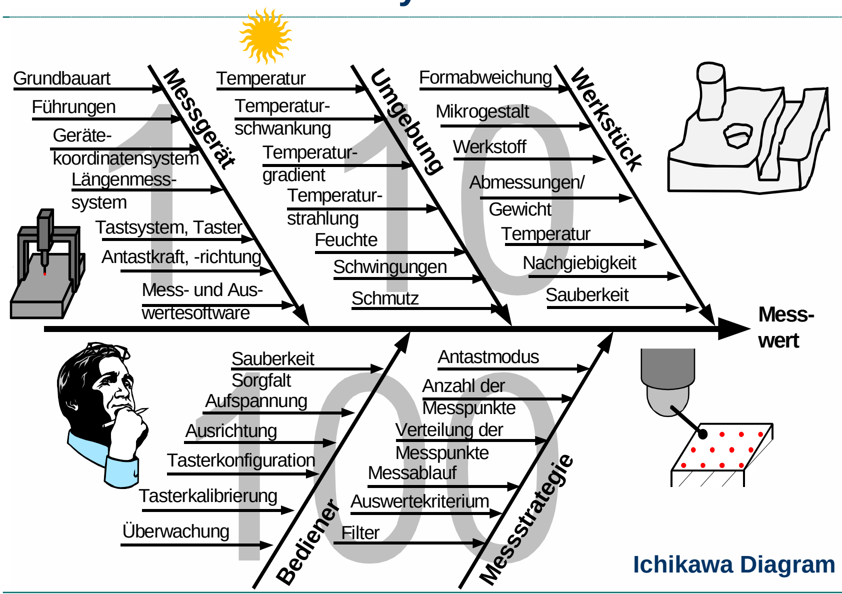
Sources of uncertainty in measurements

2



Ichikawa Diagram

Source: Bernd R.L. Siebert

Modelling the measurement

2

GUM 4.1.1

In most cases, a measurand Y is not measured directly, but is determined from N other quantities $X_1, X_2, \ldots, X_N$ through a functional relationship f :

$Y = f(X_1, X_2, \ldots, X_N)$ (GUM 1)

NOTES:

A) Quantities are denoted by capital letters, their estimates by lower case letters, i.e. the estimate of X_i (strictly speaking, of its expectation) is denoted by x_i.

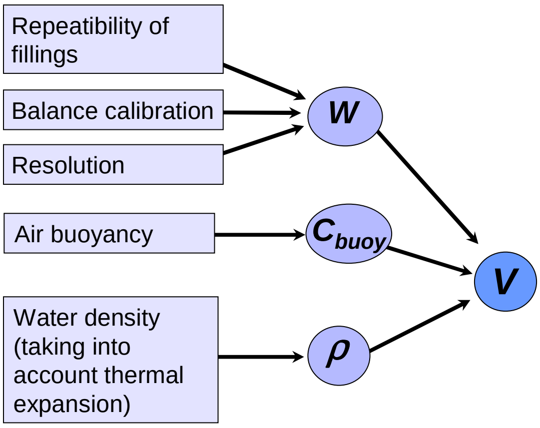
B) In a series of observations, the k_{th} observed value of X_i is denoted by $X_{i,k}$
e.g., if R denotes the resistance of a resistor, the k_{th} observed value of the R is denoted by R_k .

C) f is considered to be a function which contains every quantity, including all corrections and correction factors, that can contribute a significant component of uncertainty to the measurement result (GUM 4.1.2)

Example to be developed:

Calibration of a volume

2



$$V = \frac{m}{\rho} = \frac{W + C_{buoy}}{\rho}$$

V	Volume of the flask
m	Mass of the filled water
W	Weighing result corrected by mass of the flask
C_{buoy}	Correction by air buoyancy
ρ	Density of the water

$$V = \frac{m}{\rho} = \frac{\overline{W} + \delta W_{rep} + \Delta W_{cal} + \delta W_{cal} + \delta W_{res} + C_{buoy} + \delta C_{buoy}}{\rho + \delta\rho}$$

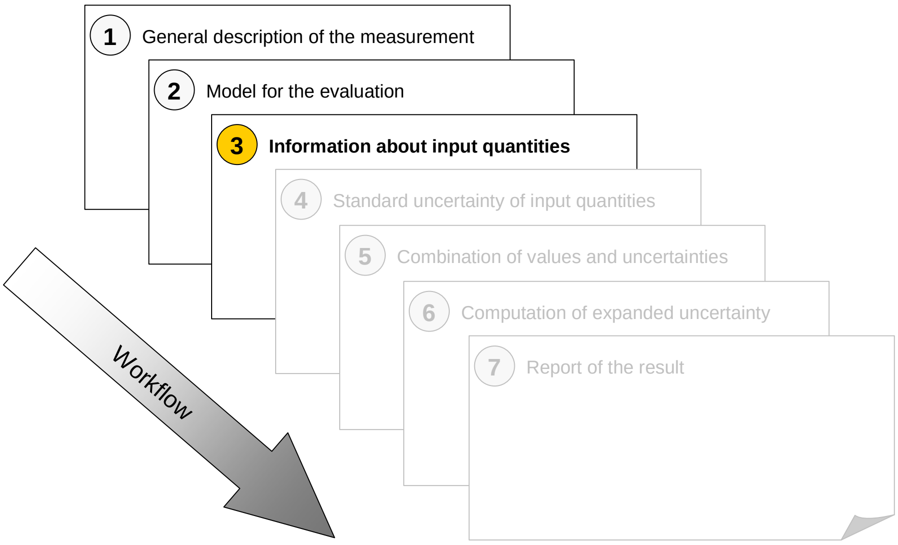
Example to be developed:

Calibration of a volume

2

Template for the uncertainty budget

i	Quantity	Value x_i	Uncertainty	Distribution	d.f. ν_i	Standard Uncertainty $u(x_i)$	Sensitivity Coefficient c_i	Uncertainty Contribution c_i*u_i	"Index" (Variance) $(c_i*u_i/u_c)^2$
1	**Weighing result: *W***								
1c	Repeatability: $\overline{W}$, δW_{rep}								
1a	Calibration: ΔW_{cal}, δW_{cal}								
1b	Resolution: δW_{res}								
2	**Air buoyancy: C_{buoy}, δC_{buoy}**								
3	**Water density: ρ**								
	Volumen: *V*								



Information about input quantities

3

Quantify all input quantities via measurements of other methods

Result of step 3:

Quantitative information on each input quantity

- Best estimate
- Dispersion, distribution

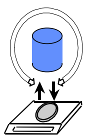
C 3 - 25

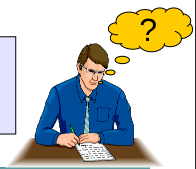
Evaluation Methods: Type A and Type B

3

The best estimates x_i for the input quantities X_i and the associated uncertainties $u(x_i)$ are determined with 2 different methods:

Type A	Type B
➨ Repeated measurements under the same conditions of measurement	➨ Other sources of information • Previous measurements • Instrument manuals • Calibration certificates • Experience, general knowledge • etc.
Information obtained during the measurement process	Use of information “external” to the measurement process





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C 3 - 26

Evaluation Methods: Type A and Type B

3

GUM 3.3.4

The purpose of the Type A and Type B classification is to indicate the two different ways of evaluating uncertainty components and is for convenience of discussion only;

the classification is **not** meant to indicate that there is **any difference in the nature** of the components resulting from the two types of evaluation.

Both types of evaluation are based on probability distributions, and the uncertainty components resulting from either type are quantified by variances or standard deviations.

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Type A evaluation: Best estimate

3

GUM 4.2.1

In most cases, the best available estimate of the expectation or expected value of a quantity q that varies randomly (random variable), and for which n independent observations q_k have been obtained under the same conditions of measurement is the arithmetic mean or average $\bar{q}$ of the n observations:

$$\bar{q} = \frac{1}{n}\sum_{k=1}^{n} q_k \qquad \text{GUM (3)}$$

Type A evaluation: Dispersion

3

GUM 4.2.2

The individual observations q_k differ in value because of random variations in the influence quantities, or random effects. The experimental variance of the observations, which estimates the variance σ^2 of the probability distribution of q, is given by:

$$s^2(q_k) = \frac{1}{n-1}\sum_{j=1}^{n} \left(q_j - \bar{q}\right)^2 \qquad \text{GUM (4)}$$

$s(q_k)$ **experimental standard deviation**

characterizes the dispersion of the observed values q_k about their mean $\bar{q}$

C 3 - 29

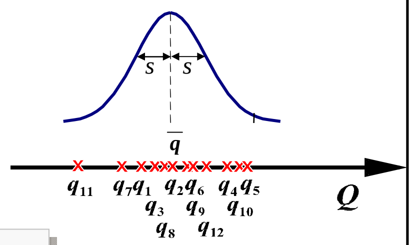
Type A evaluation: Measurement uncertainty

3

Experimental Standard Deviation of the Mean: $s(\bar{q}) = \dfrac{s(q_k)}{\sqrt{n}}$

$$u_A(\bar{q}) = s(\bar{q}) = \sqrt{\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\left(q_j - \bar{q}\right)^2}$$

n number of observations
q_j result of observations j
$\bar{q}$ mean of the n observations
s experimental standard deviation



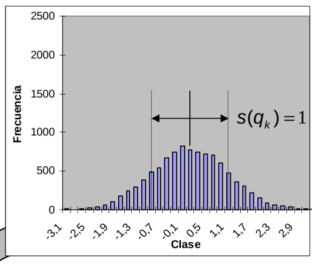
GUM 4.2.3:

The **experimental standard deviation of the mean** $s(\bar{q})$ quantifies how well $\bar{q}$ estimates the expectation of q, and may be used as a **measure of the uncertainty of** $\bar{q}$.

C 3 - 30

Standard deviation of the mean

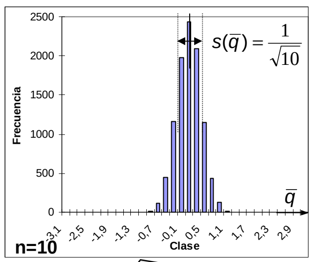
3

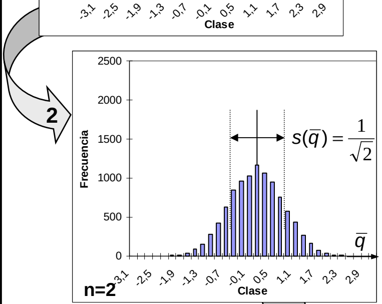


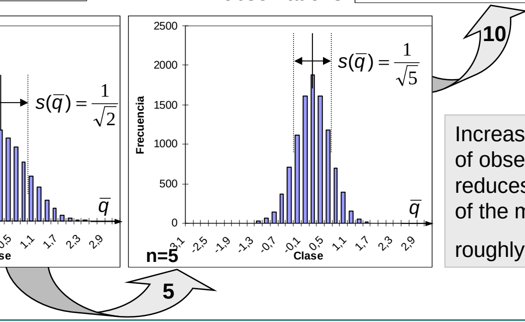
Individual observations

$s(\bar{q}) = \dfrac{s(q_k)}{\sqrt{n}}$

Mean of n = 2, 5, 10 observations







Increasing the number of observations n reduces the uncertainty of the mean $u(\bar{q})$ roughly by $1/\sqrt{n}$

Type A evaluation: Some remarks

1) The distribution of the mean $\bar{q}$ generally can be considered as Gaussian (approximately)

2) Increasing the number of observations n reduces the uncertainty of the mean $u(\bar{q})$ roughly by $1/\sqrt{n}$

3) The number of observations n should be large enough to ensure that $\bar{q}$ provides a reliable estimate of the value of the input quantity and that $s(\bar{q})$ provides a reliable estimate of its uncertainty (i.e. its standard deviation $\sigma(\bar{q})$)

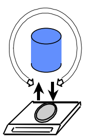
4) The difference between $s(\bar{q})$ and $\sigma(\bar{q})$ must be considered when one constructs confidence intervals. In this case, if the probability distribution of $\bar{q}$ is a normal distribution, the difference is taken into account through the *t*-distribution with $v = n$-1 degrees of freedom.

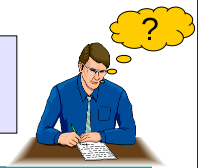
5) For a well-characterized measurement under statistical control, a pooled experimental standard deviation s_p that characterizes the measurement may be available. In such cases, when the value of a measurand q is determined from n independent observations, the experimental standard deviation of $\bar{q}$ is estimated better by $s_p/\sqrt{n}$ than by $s(q_k)/\sqrt{n}$ and the uncertainty is $u(\bar{q}) = s_p/\sqrt{n}$

Evaluation Methods: Type A and Type B

3

Type A	Type B
➨ Repeated measurements	➨ Other sources of information
under the same conditions of measurement	• Previous measurements • Instrument manuals • Calibration certificates • Experience, general knowledge • etc.
Information obtained during the measurement process	Use of information “external” to the measurement process





PTB dpm
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Distributions

3

Available information	Assigned PDF and illustration (not to scale)
Lower and upper limits a, b	Rectangular: $\mathrm{R}(a, b)$
Sum of two quantities assigned rectangular distributions with lower and upper limits a_1, b_1 and a_2, b_2	Trapezoidal: $\mathrm{Trap}(a, b, \beta)$ with $a = a_1 + a_2$, $b = b_1 + b_2$, $\beta = \lvert(b_1 - a_1) - (b_2 - a_2)\rvert/(b - a)$
Sum of two quantities assigned rectangular distributions with lower and upper limits a_1, b_1 and a_2, b_2 and the same semi-width ($b_1 - a_1 = b_2 - a_2$)	Triangular: $\mathrm{T}(a, b)$ with $a = a_1 + a_2$, $b = b_1 + b_2$
Sinusoidal cycling between lower and upper limits a, b	Arc sine (U-shaped): $\mathrm{U}(a, b)$
Best estimate x and associated standard uncertainty $u(x)$	Gaussian: $\mathrm{N}(x, u^2(x))$
Best estimate x, expanded uncertainty U_p, coverage factor k_p and effective degrees of freedom ν_{eff}	Scaled and shifted t: $t_{\nu_{\mathrm{eff}}}(x, (U_p/k_p)^2)$

Source: GUM-S1

← Verified instrument (Rectangular)

← Calibrated instrument (Gaussian)

← Calibrated instrument with additional information in the certificate (Scaled and shifted t)

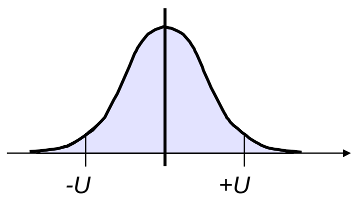
Type B evaluation: Examples

3

Calibration certificate:

➔ Normal distribution (generally)

- Expanded uncertainty U
- Level of confidence p
- Coverage factor k_p

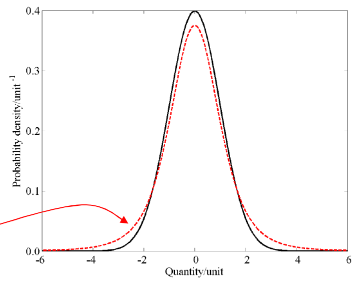


➔ t-distribution (low number of repeated measurements)

- (effective) degrees of freedom v_{eff}
- t_p (v) replaces k_p

Table G.2 — Value of $t_p(\nu)$ from the t-distribution for degrees of freedom ν that defines an interval $-t_p(\nu)$ to $+t_p(\nu)$ that encompasses the fraction p of the distribution

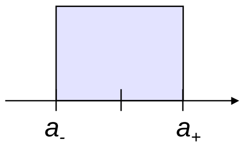
Degrees of freedom	Fraction p in percent					
ν	68,27 a)	90	95	95,45 a)	99	99,73 a)
1	1,84	6,31	12,71	13,97	63,66	235,80
2	1,32	2,92	4,30	4,53	9,92	19,21
3	1,20	2,35	3,18	3,31	5,84	9,22
4	1,14	2,13	2,78	2,87	4,60	6,62
5	1,11	2,02	2,57	2,65	4,03	5,51



Type B evaluation: Examples

Lower and upper limit of a parameter:

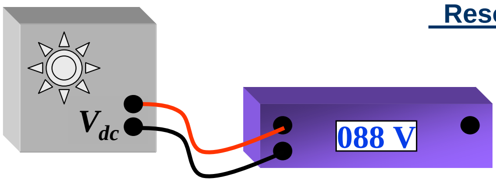
➔ Rectangular distribution



Examples:

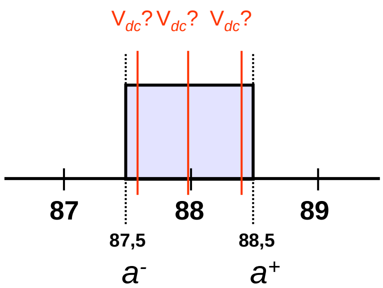
- Manufacturers specifications, MPE
- Verification of en instrument (legal metrology)
- Control of environmental parameters
- etc.

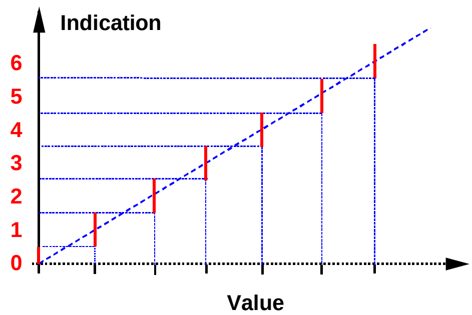
Type B evaluation



Resolution of a digital instrument:

➔ Rectangular distribution





Example to be developed:

Calibration of a volume

3

$$V = \frac{m}{\rho} = \frac{\overline{W} + \delta W_{rep} + \Delta W_{cal} + \delta W_{cal} + \delta W_{res} + C_{buoy} + \delta C_{buoy}}{\rho + \delta\rho}$$

- Repeated fillings: W = 1 992 g
  1 993 g
  1 990 g
  1 996 g
  1 994 g

$$\overline{w} = \frac{1}{5}\sum_{j=1}^{5} w_j = 1993{,}0\ g$$

$$u_A(\overline{w}) = \sqrt{\frac{1}{5\cdot 4}\sum_{j=1}^{5}\left(w_j - \overline{w}\right)^2} = 1{,}0\ g$$

- Balance calibration → Certificate: Deviation = 0,2 g
  Uncertainty U = 1,2 g (k=2)
- Balance resolution → 1 g
- Water density: t = (20 ± 2) ºC
- Air buoyancy: t_a = (20 ± 2) ºC / p_a = 1014 hPa ± ? / h_r = 50% ± ?

Example to be developed:

Calibration of a volume

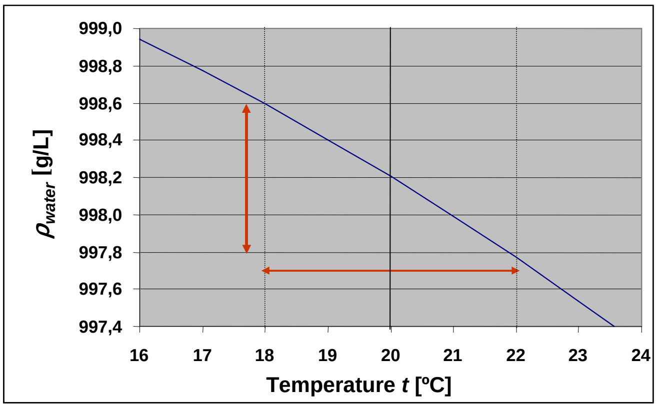
3

Water density:

$$\rho_{water} = 999{,}974\,95\ \frac{g}{L}\cdot\left[1 - \frac{\left(t - 3{,}983\,035^{\circ}C\right)^2\cdot\left(t + 301{,}797^{\circ}C\right)}{552\,528{,}9(^{\circ}C)^2\cdot\left(t + 69{,}348\,81^{\circ}C\right)}\right]$$

t : Temperature in ºC.

Tanaka et al Metrologia **38** (2001) p. 301 - 309



- Water density: t = (20 ± 2) ºC → ρ_{water} = (998,2 ± 0,4) g/L

Example to be developed:

Calibration of a volume

3

Air density:

$$\rho_{air} = \frac{p_a \cdot 0{,}348\,44\ {}^\circ C / hPa + h_r \cdot (0{,}020\,582\ {}^\circ C - t_a \cdot 0{,}002\,52)}{t_a + 273{,}15\ {}^\circ C}\ g/L$$

According to EURAMET Calibration Guide 19, eq. (4)

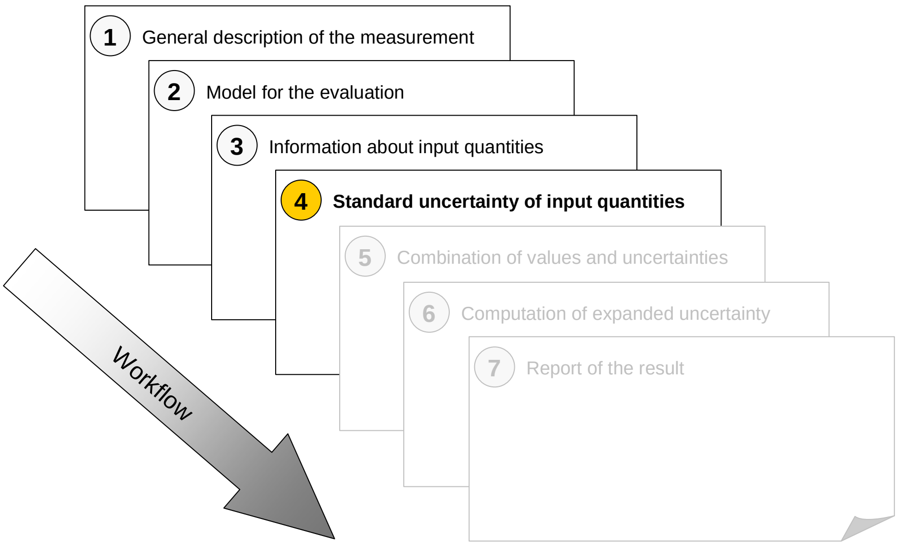
p_a Air pressure
t_a Air temperature
h_r Relative humidity

Air density: t_a = 20 ºC / p_a = 1014 hPa / h_r = 50% → ρ_{air} = 1,20 g/L

Within the “normal” variations of the atmosphere conditions the air density should not vary by more than ± 4 or 5 %

→ ρ_{air} = (1,20 ± 0,05) g/L

$C_{buoy} = \rho_{air} * V$ → C_{buoy} = (2,4 ± 0,1) g rectangular distribution



Expression of Measurement Uncertainty according to GUM
Tirana, Albania, 2 to 4 June 2010
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Standard uncertainty of input quantities

4

1) Type A evaluation:
   Has provided the uncertainty already as standard deviation
2) Type B evaluation:
   Calculate standard deviation of all input quantities from the assumed distribution

Result of step 4:

Uncertainty of all input quantities expressed as standard deviation

Standard uncertainty of input quantities

4

GUM 3.3.5

The estimated variance u^2 characterizing an uncertainty component obtained from a Type A evaluation is calculated from series of repeated observations and is the familiar statistically estimated variance s^2 (see 4.2). The estimated **standard deviation** u, the positive square root of u^2, is thus $u = s$ and for convenience is sometimes called a *Type A standard uncertainty*.

For an uncertainty component obtained from a Type B evaluation, the estimated variance u^2 is evaluated using available knowledge, and the estimated standard deviation u is sometimes called a *Type B standard uncertainty*.

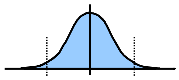
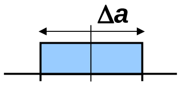
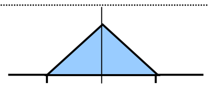
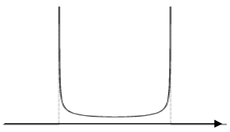
Thus a Type A standard uncertainty is obtained from a **probability density function** derived from an **observed frequency distribution**,

while a Type B standard uncertainty is obtained from an assumed probability density function based on the degree of belief that an event will occur (often called subjective **probability**).

Both approaches employ recognized interpretations of probability.

Type B: Standard uncertainty

4

Distribution		Standard uncertainty	
	Normal distribution: expanded uncertainty U	$u_x = \frac{U}{k}$	
	Rectangular distribution: lower/upper limit	$u_x = \frac{\Delta a}{\sqrt{12}} = \frac{a_+ - a_-}{\sqrt{12}}$	see GUM (6)
	Triangular distribution: lower/upper limit	$u_x = \frac{\Delta a}{\sqrt{24}} = \frac{a_+ - a_-}{\sqrt{24}}$	see GUM (9b)
	U-shaped distribution: lower/upper limit	$u_x = \frac{\Delta a}{\sqrt{8}} = \frac{a_+ - a_-}{\sqrt{8}}$	

Example to be developed:

Calibration of a volume

4

Determine the standard uncertainties

- Repeatability: u_A = 1,0 g
- Balance calibration: U = 1,2 g (k=2) → u = 0,6 g
- Balance resolution: 1 g → $u = \frac{1\,g}{\sqrt{12}} = 0{,}29\,g$
- Water density: ± 0,4 g/L → $u = \frac{0{,}8\,g/L}{\sqrt{12}} = 0{,}23\,g$
- Air buoyancy: ± 0,1 g → $u = \frac{0{,}2\,g}{\sqrt{12}} = 0{,}06\,g$

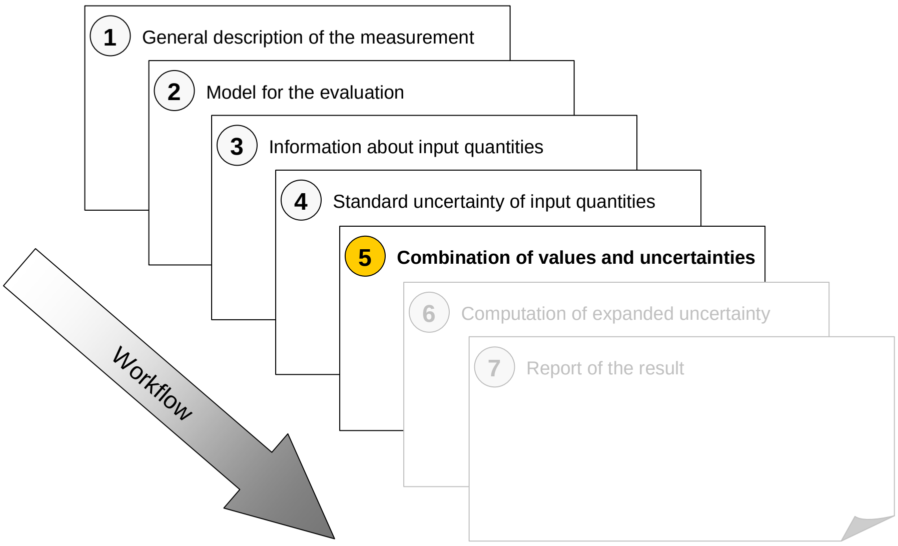
Example to be developed:

Calibration of a volume

4

Determine the standard uncertainties

i	Quantity	Value x_i	Uncertainty	Distribution	Standard Uncertainty $u(x_i)$
1	**Weighing result: *W***	***1992,8 g***			
1c	Repeatability: $\hat{W}$, δW_{rep}	1993,0 g	1,0 g	normal	1,0 g
1a	Calibration: ΔW_{cal}, δW_{cal}	-0,2 g	1,2 g	normal, k=2	0,60 g
1b	Resolution: δW_{res}	0,0 g	1,0 g	rectangular	0,29 g
2	**Air buoyancy: C_{buoy}, δC_{buoy}**	***2,4 g***	+/- 0,1 g	rectangular	0,06 g
3	**Water density: ρ**	***998,2 g/L***	+/- 0,4 g/L	rectangular	0,23 g/L
	Volumen: *V*	**1,9988 L**			



Combination of values and uncertainties

5

1) Calculate the value of the measurand
2) Determine the sensitivity coefficient of each input quantity
3) Analyze correlations between input quantities
4) Calculate the standard uncertainty of the measurand using the "Law of propagation of uncertainties"

Result of step 5:

- Best estimate of the measurand
- Combined standard uncertainty of the measurand

Value of the measurand

5

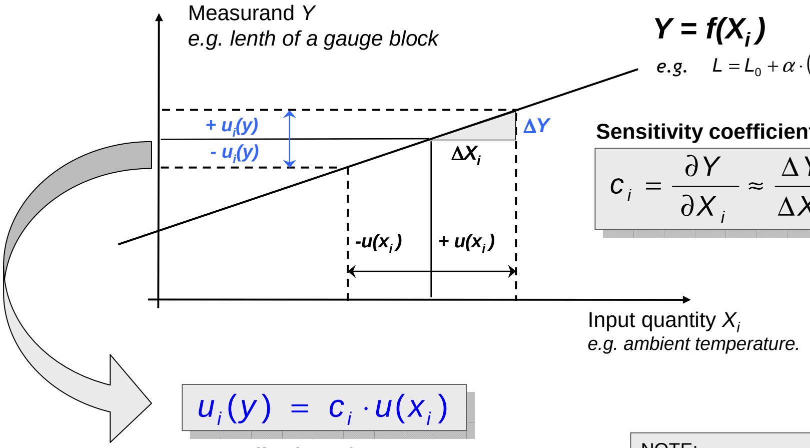
Estimate of measurand Y :

$$y = f(x_1, x_2, \ldots, x_N)$$

x_k best estimate of input quantity X_k

Sensitivity coefficient

5



$Y = f(X_i)$

e.g. $L = L_0 + \alpha \cdot (T - T_0)$

Sensitivity coefficient

$$c_i = \frac{\partial Y}{\partial X_i} \approx \frac{\Delta Y}{\Delta X_i}$$

$$u_i(y) = c_i \cdot u(x_i)$$

Contribution of X_i to the uncertainty of Y

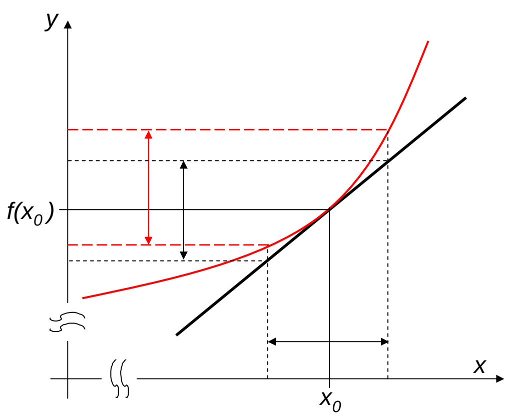
NOTE:
Sensitivity coefficients can also be determined experimentally (in some cases)

Sensitivity coefficient

5

Taylor expansion of *f(x)* in interval of variation of *x*

$$f(x) = f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0} \cdot \delta x + \frac{1}{2} \cdot \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0} \cdot \delta x^2 + \ldots$$



Consideration of higher order terms might be required

Alternative: GUM-S1 (MCS)

Law of propagation of uncertainty

5

in the case of independent (uncorrelated) input quantities

$$u_c(y) = \sqrt{\sum_i \left[c_i \cdot u(x_i)\right]^2} = \sqrt{\sum_i \left[\frac{\partial Y}{\partial X_i} \cdot u(x_i)\right]^2}$$

GUM (10)

$u_c(y)$	Combined standard uncertainy of Y
$u(x_i)$	Standard uncertainty of the input quantity X_i
$c_i = \frac{\partial Y}{\partial X_i}$	Sensitivity coefficient of of the input quantity X_i

NOTE When the nonlinearity of f is significant, higher-order terms in the Taylor series expansion must be included in the expression for $u_c^2(y)$, Equation (10). When the distribution of each X_i is normal, the most important terms of next highest order to be added to the terms of Equation (10) are

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\frac{1}{2}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2 + \frac{\partial f}{\partial x_i}\frac{\partial^3 f}{\partial x_i \partial x_j^2}\right]u^2(x_i)u^2(x_j)$$

Correlated input quantities

Examples:

- two or more input quantities are influenced by the same effect
  e.g. electrical resistance R = U / I

- use of the same standard for the measurement of two or more input quantities

- one input quantity depends directly on another input quantity
  e.g. atmospheric pressure, ambient temperature

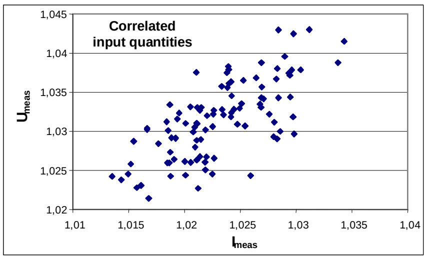
Uncertainty might

(a) increase or (b) decrease

if the random errors of the correlated input quantities …

a) … contribute to the measurand in the same sense

b) … compensate themselves partially



Correlation

Law of propagation of uncertainty for correlated input quantities

see GUM 5.2.2

5.2.2 When the input quantities are correlated, the appropriate expression for the combined variance $u_c^2(y)$ associated with the result of a measurement is

$$u_c^2(y) = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} u(x_i, x_j) = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} u(x_i, x_j) \qquad (13)$$

where x_i and x_j are the estimates of X_i and X_j and $u(x_i, x_j) = u(x_j, x_i)$ is the estimated covariance associated with x_i and x_j. The degree of correlation between x_i and x_j is characterized by the estimated **correlation coefficient** (C.3.6)

$$r(x_i, x_j) = \frac{u(x_i, x_j)}{u(x_i)u(x_j)} \qquad (14)$$

Alternative: GUM-S1 (MCS)

How to estimate the correlation coefficient r(x_i,x_j) ?

see GUM 5.2.3

Example to be developed:

Calibration of a volume

5

Determine best estimate of the measurand

$$V = \frac{\overline{W} + \Delta W_{cal} + C_{buoy}}{\rho}$$

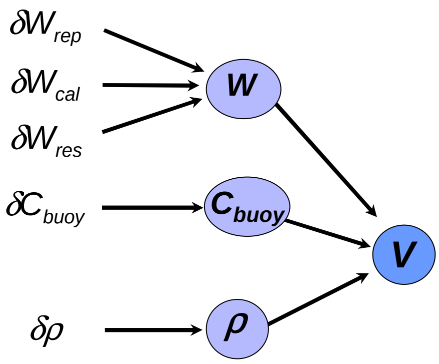
$$V = \frac{1993{,}0\,g - 0{,}2\,g + 2{,}4\,g}{998{,}2\,g/L} = 1{,}9988\,L$$

Example to be developed:

Calibration of a volume

5

$$V = \frac{W + C_{buoy}}{\rho}$$



$$c_W = \frac{\partial V}{\partial W} = \frac{1}{\rho}$$

$$c_{C-buoy} = \frac{\partial V}{\partial C_{buoy}} = \frac{1}{\rho}$$

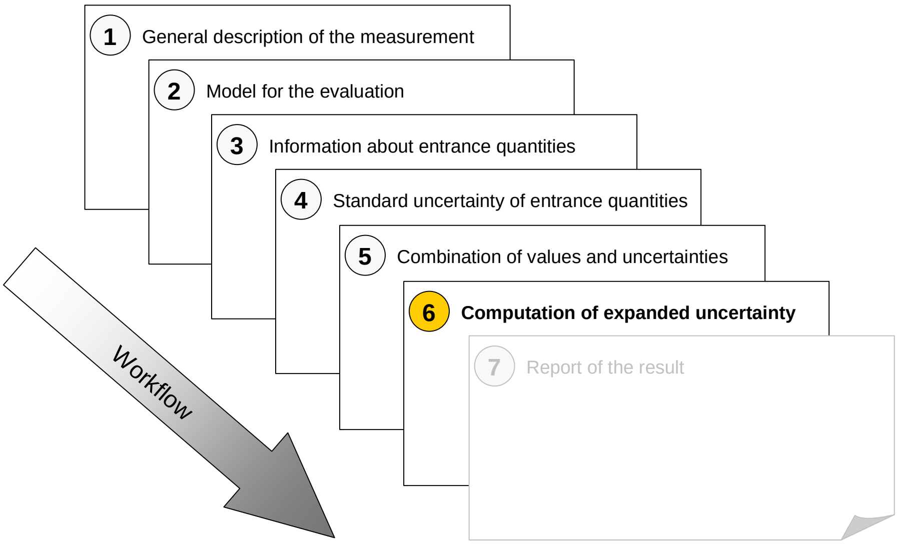
$$c_\rho = \frac{\partial V}{\partial \rho} = -\frac{W + C_{buoy}}{\rho^2}$$

Example to be developed:

Calibration of a volume

5

i	Quantity	Value x_i	Uncertainty	Distribution	Standard Uncertainty $u(x_i)$	Sensitivity Coefficient c_i	Uncertainty Contribution $c_i{*}u_i$	"Index" (Variance) $(c_i{*}u_i/u_c)^2$
1	**Weighing result: *W***	***1992,8 g***						
1c	Repeatability: $\hat{W}$, δW_{rep}	1993,0 g	1,0 g	normal	1,0 g	0,00100 L/g	***0,0010 L***	***60,3%***
1a	Calibration: ΔW_{cal}, δW_{cal}	-0,2 g	1,2 g	normal, k=2	0,60 g	0,00100 L/g	***0,0006 L***	***21,7%***
1b	Resolution: δW_{res}	0,0 g	1,0 g	rectangular	0,29 g	0,00100 L/g	***0,0003 L***	***5,0%***
2	**Air buoyancy: C_{buoy}, δC_{buoy}**	***2,4 g***	+/- 0,1 g	rectangular	0,06 g	0,00100 L/g	***0,0001 L***	***0,2%***
3	**Water density: ρ**	***998,2 g/L***	+/- 0,4 g/L	rectangular	0,23 g/L	0,00200 L²/g	***0,0005 L***	***12,8%***
	Volumen: *V*	**1,9988 L**				***u_c =***	**0,0013 L**	



Expanded uncertainty

6

GUM 6.1.2

Although $u_c(y)$ can be universally used to express the uncertainty of a measurement result, in some commercial, industrial, and regulatory applications, and when health and safety are concerned, it is often necessary to give a measure of uncertainty that defines an interval about the measurement result that may be expected to encompass a large fraction of the distribution of values that could reasonably be attributed to the measurand.

GUM 6.2.1

The additional measure of uncertainty that meets the requirement of providing an interval of the kind indicated in 6.1.2 is termed *expanded uncertainty* and is denoted by U. The expanded uncertainty U is obtained by multiplying the combined standard uncertainty $uc(y)$ by a *coverage factor* k:

$$U = k\, u_c(y) \qquad (18)$$

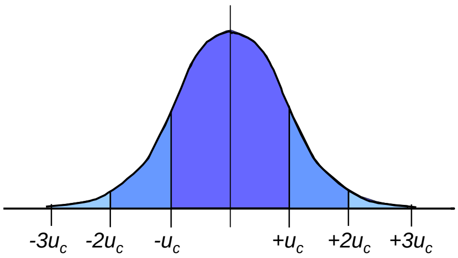
The result of a measurement is then conveniently expressed as $Y = y \pm U$, which is interpreted to mean that the best estimate of the value attributable to the measurand Y is y, and that $y - U$ to $y + U$ is an interval that may be expected to encompass a large fraction of the distribution of values that could reasonably be attributed to Y.

Expanded uncertainty

6

Central Limit Theorem:

The distribution of the measurand Y will be approximately normal,
if the input quantities X_i are independent (no correlation)
and the variance of $s^2(Y)$ is much larger
than any single component $c_i^2 \cdot s^2(X_i)$ from a non normaly distributed X_i.

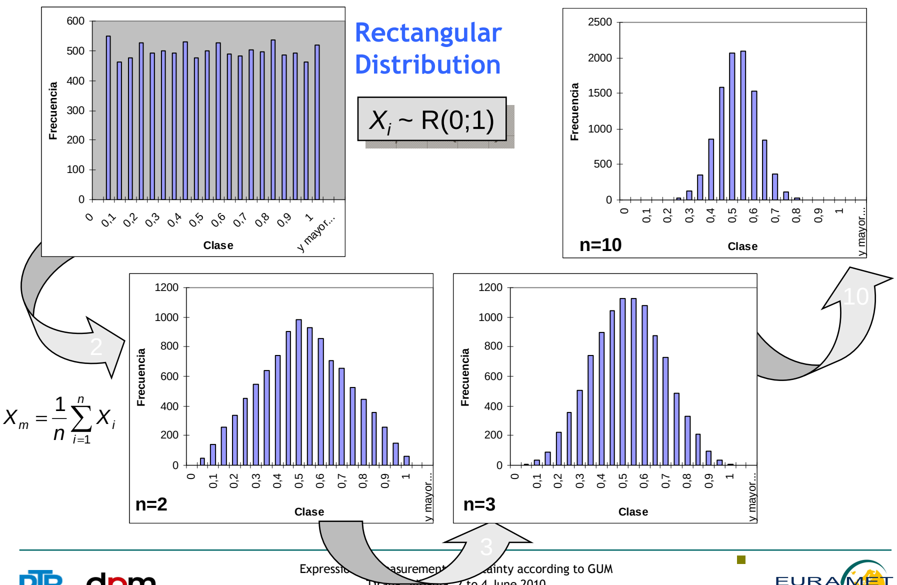


Expanded Uncertainty: $U = k \cdot u_c$

Coverage Factor	k	1	2	3
Level of Confidence	p	68,3 %	95,4 %	99,7 %

Example: Distribution of the mean

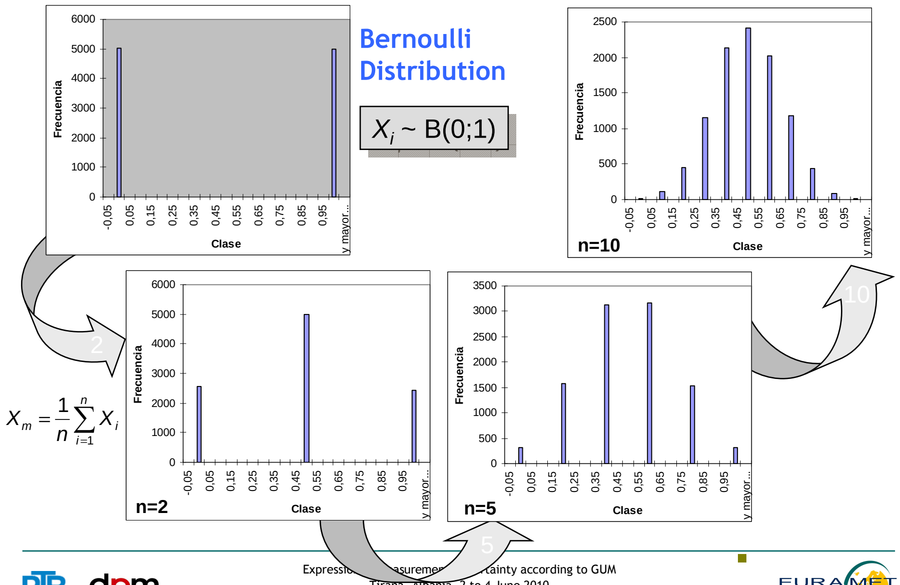
6



$$X_m = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Example: Distribution of the mean

6



Degrees of freedom

6

Frequently, a type A evaluation is based on a small number n of observations.

How reliable is it to base the estimation of the measurement uncertainty $u_A(\bar{x})$ on the experimental standard deviation

$$s(\bar{x}) = \sqrt{\frac{1}{n} \cdot \frac{1}{n-1} \sum_{i=1}^{n} (\bar{x} - x_i)^2}$$

obtained with a small number of observations?

A more reliable way to estimate the expanded uncertainty U_p is replacing

$$U_p = k \cdot u_A(\bar{x}) \quad \text{by} \quad U_p = t_p(\nu) \cdot u_A(\bar{x})$$

p coverage probability
$\nu = n - 1$ degrees of freedom

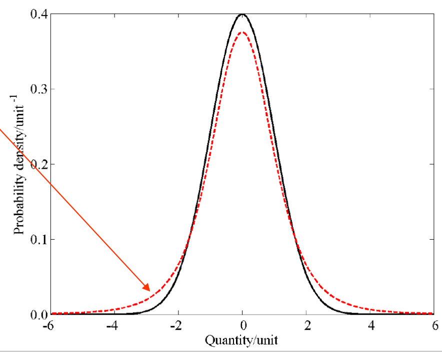
t-distribution (Student´s distribution)

6

Table G.2 — Value of $t_p(\nu)$ from the t-distribution for degrees of freedom ν that defines an interval $-t_p(\nu)$ to $+t_p(\nu)$ that encompasses the fraction p of the distribution

Degrees of freedom	Fraction p in percent					
ν	68,27 [a)]	90	95	95,45 [a)]	99	99,73 [a)]
1	1,84	6,31	12,71	13,97	63,66	235,80
2	1,32	2,92	4,30	4,53	9,92	19,21
3	1,20	2,35	3,18	3,31	5,84	9,22
4	1,14	2,13	2,78	2,87	4,60	6,62
5	1,11	2,02	2,57	2,65	4,03	5,51
6	1,09	1,94	2,45	2,52	3,71	4,90
7	1,08	1,89	2,36	2,43	3,50	4,53
8	1,07	1,86	2,31	2,37	3,36	4,28
9	1,06	1,83	2,26	2,32	3,25	4,09
10	1,05	1,81	2,23	2,28	3,17	3,96
11	1,05	1,80	2,20	2,25	3,11	3,85
12	1,04	1,78	2,18	2,23	3,05	3,76
13	1,04	1,77	2,16	2,21	3,01	3,69
14	1,04	1,76	2,14	2,20	2,98	3,64
15	1,03	1,75	2,13	2,18	2,95	3,59
16	1,03	1,75	2,12	2,17	2,92	3,54
17	1,03	1,74	2,11	2,16	2,90	3,51
18	1,03	1,73	2,10	2,15	2,88	3,48
19	1,03	1,73	2,09	2,14	2,86	3,45
20	1,03	1,72	2,09	2,13	2,85	3,42
25	1,02	1,71	2,06	2,11	2,79	3,33
30	1,02	1,70	2,04	2,09	2,75	3,27
35	1,01	1,70	2,03	2,07	2,72	3,23
40	1,01	1,68	2,02	2,06	2,70	3,20
45	1,01	1,68	2,01	2,06	2,69	3,18
50	1,01	1,68	2,01	2,05	2,68	3,16
100	1,005	1,660	1,984	2,025	2,626	3,077
∞	1,000	1,645	1,960	2,000	2,576	3,000

a) For a quantity z described by a normal distribution with expectation μ_z and standard deviation σ, the interval $\mu_z \pm k\sigma$ encompasses p = 68,27 percent, 95,45 percent and 99,73 percent of the distribution for k = 1, 2 and 3, respectively.



GUM G.3.2

If z is a normally distributed random variable with expectation μ_z and standard deviation σ_z and z is the arithmetic mean of n independent observations z_k of z with $s(\bar{z})$ the experimental standard deviation of $\bar{z}$ then the distribution of the variable $t = (\bar{z} - \mu_z)/s(\bar{z})$ is the t-distribution … with $\nu = n$ - 1 degrees of freedom

Effective Degrees of Freedom

6

Measurand: $Y = f(X_1, X_2, \ldots, X_N)$

Combined uncertainty:
(without correlations)

$$u_c^2(y) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} \cdot u(x_i) \right)^2$$

Expanded uncertainty: $U = t_p(\nu_{ef}) \cdot u_c(y)$

Effective degrees of freedom
(Welch-Satterthwaite)

$$\frac{1}{\nu_{ef}} = \sum_{i=1}^{N} \left(\frac{u_i}{u_c} \right)^4 \cdot \frac{1}{\nu_i}$$

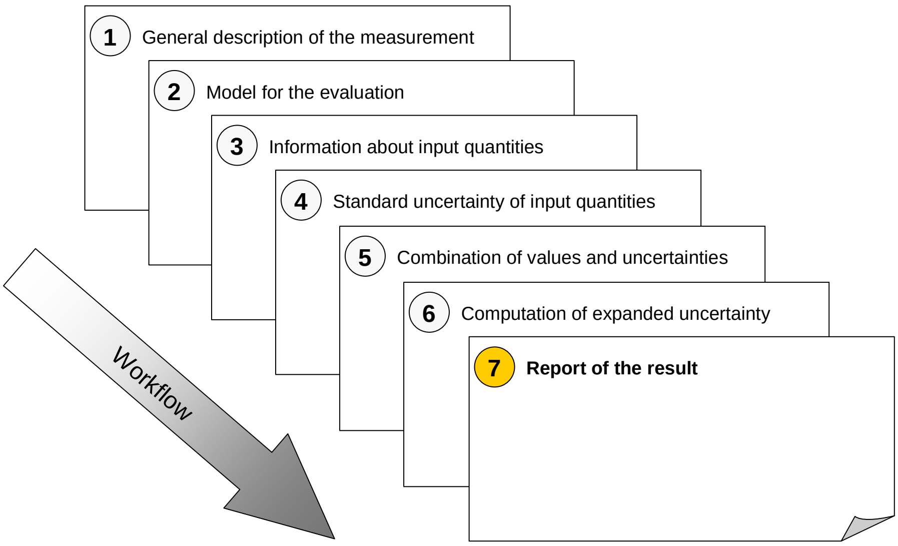
u_i Contribution of X_i to the combined uncertainty
ν_i Degrees of freedom of X_i
u_c Combined standard uncertainty of Y

Example to be developed:

Calibration of a volume

6

i	Quantity	Value x_i	Uncertainty	Distribution	Standard Uncertainty $u(x_i)$	Sensitivity Coefficient c_i	Uncertainty Contribution c_i*u_i	"Index" (Variance) $(c_i*u_i/u_c)^2$	Degrees of freedom
1	**Weighing result: *W***	***1992,8 g***							
1c	Repeatibility: $\hat{W}$, δW_{rep}	1993,0 g	1,0 g	normal	1,0 g	0,00100 L/g	***0,0010 L***	***60,3%***	4
1a	Calibration: ΔW_{cal}, δW_{cal}	-0,2 g	1,2 g	normal, k=2	0,60 g	0,00100 L/g	***0,0006 L***	***21,7%***	10000
1b	Resolution: δW_{res}	0,0 g	1,0 g	rectangular	0,29 g	0,00100 L/g	***0,0003 L***	***5,0%***	10000
2	**Air buoyancy: C_{buoy}, δC_{buoy}**	***2,4 g***	+/- 0,1 g	rectangular	0,06 g	0,00100 L/g	***0,0001 L***	***0,2%***	10000
3	**Water density: ρ**	***998,2 g/L***	+/- 0,4 g/L	rectangular	0,23 g/L	0,00200 L²/g	***0,0005 L***	***12,8%***	10000
	Volumen: *V*	**1,9988 L**				**u_c =**	**0,0013 L**		**11,02**
			not considering degrees of freedom			**$U_{95,45}$ =**	**0,0026 L**		
						$t_{95,45}$ =	**2,25**		
			considering degrees of freedom			**$U_{95,45}$ =**	**0,0029 L**		



Report of the results

Measurement result:

$Y = y \pm U$ with k or t = ?

Y	Measurand
y	Best estimate of the measurand
U	Expanded uncertainty

- **V = (1,998 8 ± 0,002 9) L** **(*t* = 2,25)**
- **V = 1,998 8 L ± 2,9 mL** **(*k* = 2,25)**
- **V = 1,998 8 L** ***U*** **= 2,9 mL** **(*t* = 2,25)**
- **V = 1,998 8 L** ***U*** **= 0,15 %** **(*t* = 2,25)**

Report of the results

7

GUM 7.2.7

7.2.7 In the detailed report that describes how the result of a measurement and its uncertainty were obtained, one should follow the recommendations of 7.1.4 and thus

a) give the value of each input estimate x_i and its standard uncertainty $u(x_i)$ together with a description of how they were obtained;

b) give the estimated covariances or estimated correlation coefficients (preferably both) associated with all input estimates that are correlated, and the methods used to obtain them;

c) give the degrees of freedom for the standard uncertainty of each input estimate and how it was obtained;

d) give the functional relationship $Y = f(X_1, X_2, ..., X_N)$ and, when they are deemed useful, the partial derivatives or sensitivity coefficients $\partial f/\partial x_i$. However, any such coefficients determined experimentally should be given.

Provide all information which is relevant to understand the result unambiguously

Calibration certificate

CalProf
Calibraciones Profesionales

Client name	XYZ, Ltd.
Address:	## YYY Road,
Service number:	357-2000
Cerificate number:	CP-CC-456/2000.
Calibration date:	2000–06–13
Instrument:	Glass graduated beaker
Brand:	ABC
Model:	PGV-92A-RS
Serial number	2879-1K
Standard:	Balance, *Brand, Model, Serial number*
Calibration result:	V = 1,9988 L
Uncertainty (p = 95%):	U = 0,0029 L (ν_{eff} = 11)
Environmental measurement conditions	Temperature 20 °C ±2 °C Atmospheric pressure (1014 ± 2) hPa Relative humidity (44 ± 5) %
Procedure employed:	AC-P.200 (gravimetric method)

Calibrated by José J. López *Approved by* A.K. Smith, Director Metrology *Date* 2090-06-13

Calibraciones Profesionales *Avenida Resolución #35 México D.F., C.P.12345* *Tel.: (55) 12345678*

Source: CENAM

Obtaining the measurement result in 7 steps

Key tasks:

1 **General description** of the measurement
Principle, method and procedure / relevant quantities

2 **Model** for the evaluation
X_1 ... X_N
⇨ $Y=f(X_1,..,X_N)$

3 **Information** about **input quantities**
Methods:
Type - A
Type - B
⇨ x_i , δx_i

THINK

4 Evaluate **standard uncertainties** of input quantities $u(x_i)$

5 **Combination** of values and uncertainties ⇨ y, $u(y)$

6 Computation of **expanded uncertainty** ⇨ U, k_p, (PDF ?)

FIXED RULES

7 Report the **result** ⇨ $y \pm U$, k_p, PDF
Plausible? Most relevant contributions? Improvements possible?

THINK & INTERPRET

Following an idea of Bernd R.L. Siebert