GUM Supplement 1
Propagation of distributions using a Monte Carlo method

(JCGM 101:2008)

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outline

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- probability density function (PDF)
- Monte Carlo method (MCM)
- examples

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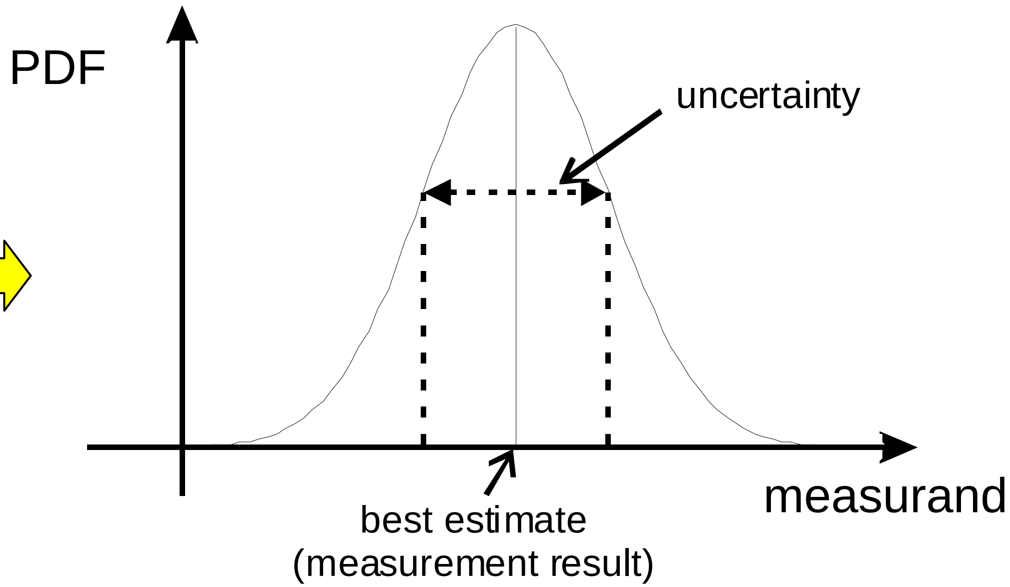
probability density function (PDF)

probability density function (PDF)

measurement data

additional information

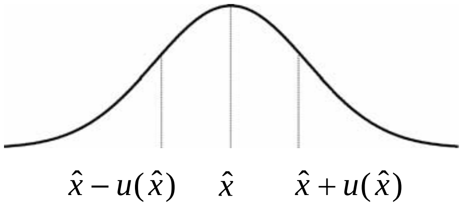


probability density function (PDF)

information ➔ PDF

$a \leq x \leq b$ ➔ rectangular

$\hat{x}, u(\hat{x})$ ➔ Gaussian

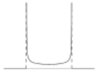
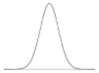
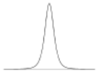
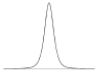


⋮

- principle of maximum entropy
- enables the consideration of systematic effects (type B)

probability density function (PDF)

Table 1 — Available information and the PDF assigned on the basis of that information (6.4.1, C.1.2)

Available information	Assigned PDF and illustration (not to scale)		Subclause
Lower and upper limits a, b	Rectangular: $\mathrm{R}(a, b)$		6.4.2
Inexact lower and upper limits $a \pm d$, $b \pm d$	Curvilinear trapezoid: $\mathrm{CTrap}(a, b, d)$		6.4.3
Sum of two quantities assigned rectangular distributions with lower and upper limits a_1, b_1 and a_2, b_2	Trapezoidal: $\mathrm{Trap}(a, b, \beta)$ with $a = a_1 + a_2$, $b = b_1 + b_2$, $\beta = \lvert(b_1 - a_1) - (b_2 - a_2)\rvert/(b - a)$		6.4.4
Sum of two quantities assigned rectangular distributions with lower and upper limits a_1, b_1 and a_2, b_2 and the same semi-width ($b_1 - a_1 = b_2 - a_2$)	Triangular: $\mathrm{T}(a, b)$ with $a = a_1 + a_2$, $b = b_1 + b_2$		6.4.5
Sinusoidal cycling between lower and upper limits a, b	Arc sine (U-shaped): $\mathrm{U}(a, b)$		6.4.6
Best estimate x and associated standard uncertainty $u(x)$	Gaussian: $\mathrm{N}(x, u^2(x))$		6.4.7
Best estimate $\boldsymbol{x}$ of vector quantity and associated uncertainty matrix $\boldsymbol{U}_{\boldsymbol{x}}$	Multivariate Gaussian: $\mathrm{N}(\boldsymbol{x}, \boldsymbol{U}_{\boldsymbol{x}})$		6.4.8
Series of indications $x_1, \ldots, x_n$ sampled independently from a quantity having a Gaussian distribution, with unknown expectation and unknown variance	Scaled and shifted t: $t_{n-1}(\bar{x}, s^2/n)$ with $\bar{x} = \sum_{i=1}^{n} x_i/n$, $s^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2/(n-1)$		6.4.9.2
Best estimate x, expanded uncertainty U_p, coverage factor k_p and effective degrees of freedom ν_{eff}	Scaled and shifted t: $t_{\nu_{\mathrm{eff}}}(x, (U_p/k_p)^2)$		6.4.9.7

Excerpt from GUM S1, table 1

probability density function (PDF) – type A

n measurement data of a Gaussian distribution

$$x_1,\ldots,x_n \qquad X_i \sim N(\mu,\sigma^2)$$

Bayes theorem ➔

$$p(\mu \mid x_1,\ldots,x_n) = \frac{1}{s/\sqrt{n}}\, t_{n-1}\left(\frac{\mu-\bar{x}}{s/\sqrt{n}}\right)$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

scaled and shifted *t* distribution with *n* -1 degrees of freedom

best estimate (*n* > 2) $\bar{x}$

uncertainty (*n* > 3) $u(\bar{x}) = \sqrt{\frac{n-1}{n-3}}\,\frac{s}{\sqrt{n}}$

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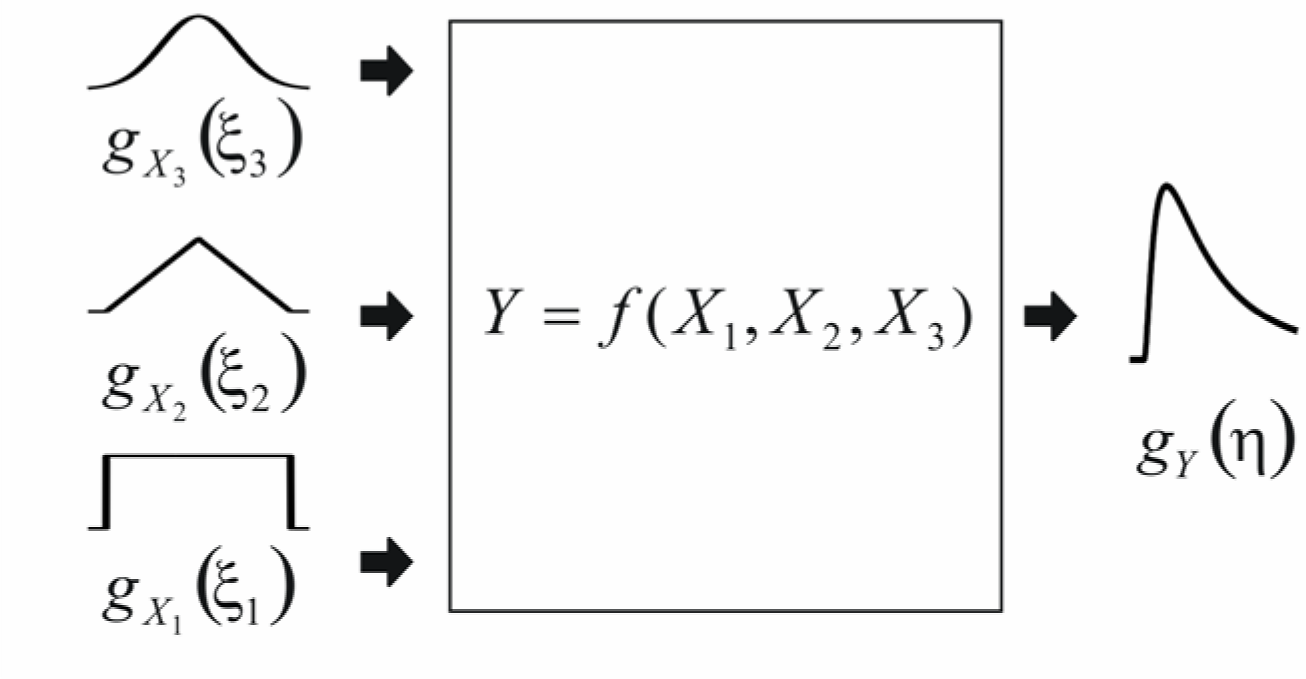
Monte Carlo method (MCM) - propagation

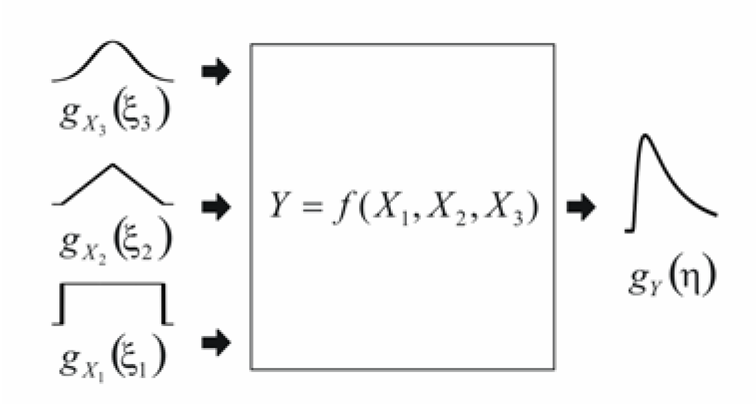
propagation of distributions

PDFs of input quantities ⇨ mathematical model ⇨ PDF of output quantities



method: transformation of random variables

Monte Carlo method (MCM) - propagation



$$g_Y(\eta) = \int g_{\mathbf{X}}(\xi)\,\delta[\eta - f(\xi)]\,\mathrm{d}\xi$$

best estimate $\quad y = \int f(\xi)\, g_{\mathbf{X}}(\xi)\,\mathrm{d}\xi$

standard uncertainty $\quad u^2(y) = \int (f(\xi) - y)^2\, g_{\mathbf{X}}(\xi)\,\mathrm{d}\xi$

coverage interval $\quad \int_a^b g_Y(\eta)\,\mathrm{d}\eta = p$

linear approximation
of the model function

$$f(\xi) \approx f(\mathbf{x}) + (\xi - \mathbf{x})^{\mathrm{T}}(\nabla f(\mathbf{x}))$$

⇩

standard GUM (GUF)

Monte Carlo method (MCM) - propagation

GUM / GUF

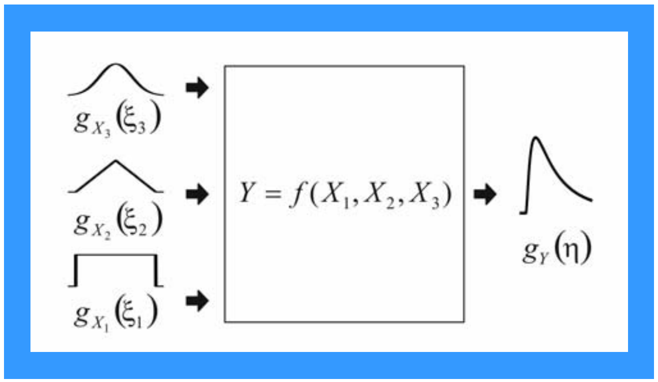
$$x_1,\ u(x_1) \rightarrow$$
$$x_2,\ u(x_2) \rightarrow \quad \boxed{Y = f(X_1, X_2, X_3)} \quad \rightarrow y,\ u(y)$$
$$x_3,\ u(x_3) \rightarrow$$

best estimate $\quad y = f(x_1, \ldots, x_N)$

standard uncertainty $\quad u^2(y) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)$

expanded uncertainty $\quad U = k\,u(y) \qquad$ coverage interval $\quad [y - U, y + U]$

Monte Carlo method (MCM) - propagation



$$g_Y(\eta) = \int g_{\mathbf{X}}(\xi)\,\delta[\eta - f(\xi)]\,\mathrm{d}\xi$$

model
$Y = f(X_1, \ldots, X_N)$

PDF of the input quantities
$g_{X_1,\ldots,X_N}(\xi_1, \ldots, \xi_N)$

$(\xi_1, \ldots, \xi_N)$ taken randomly from PDF $g_{X_1,\ldots,X_N}(\xi_1, \ldots, \xi_N)$

↓

$\eta = f(\xi_1, \ldots, \xi_N)$ mathematical model

↓

η taken randomly from PDF $g_Y(\eta)$

↓

repeated many times yields the PDF $g_Y(\eta)$

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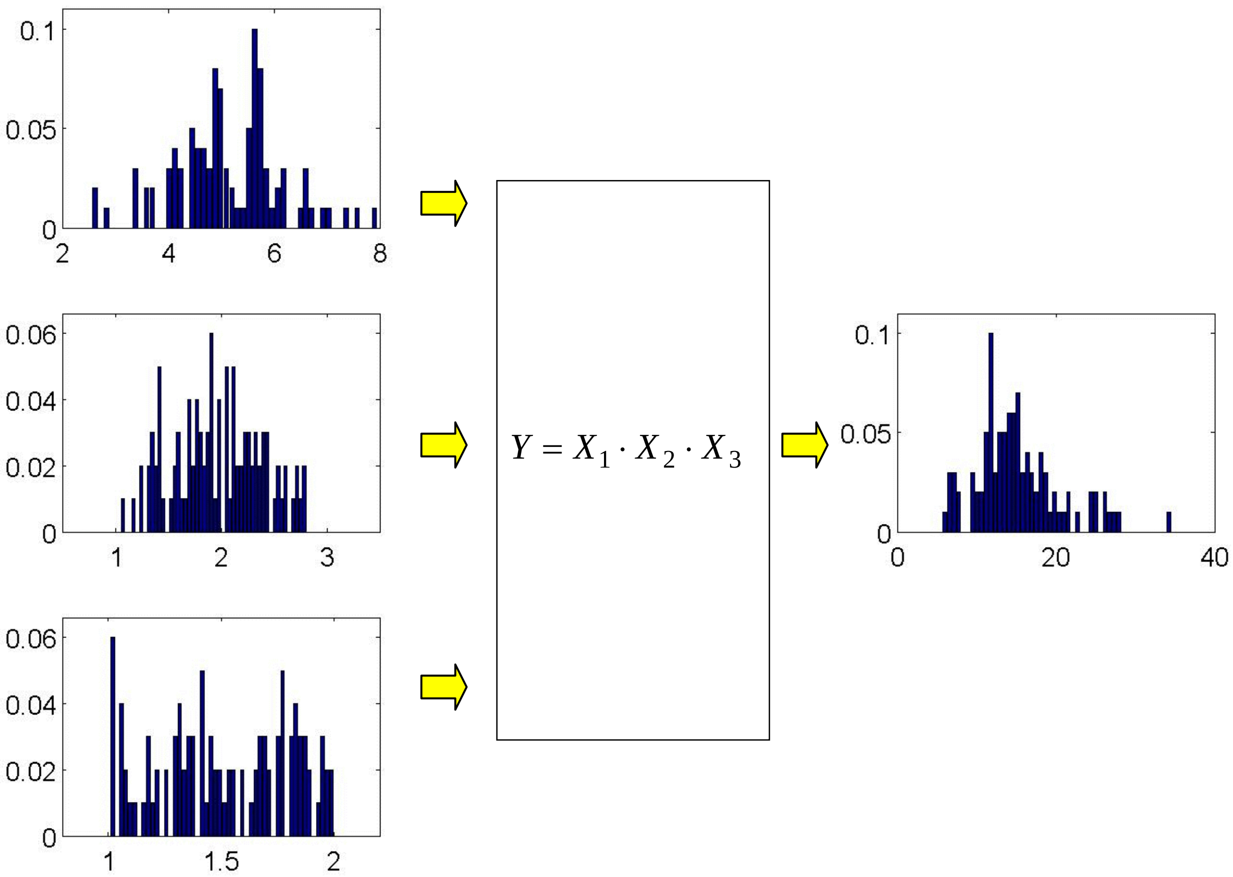
examples

10^2 trials



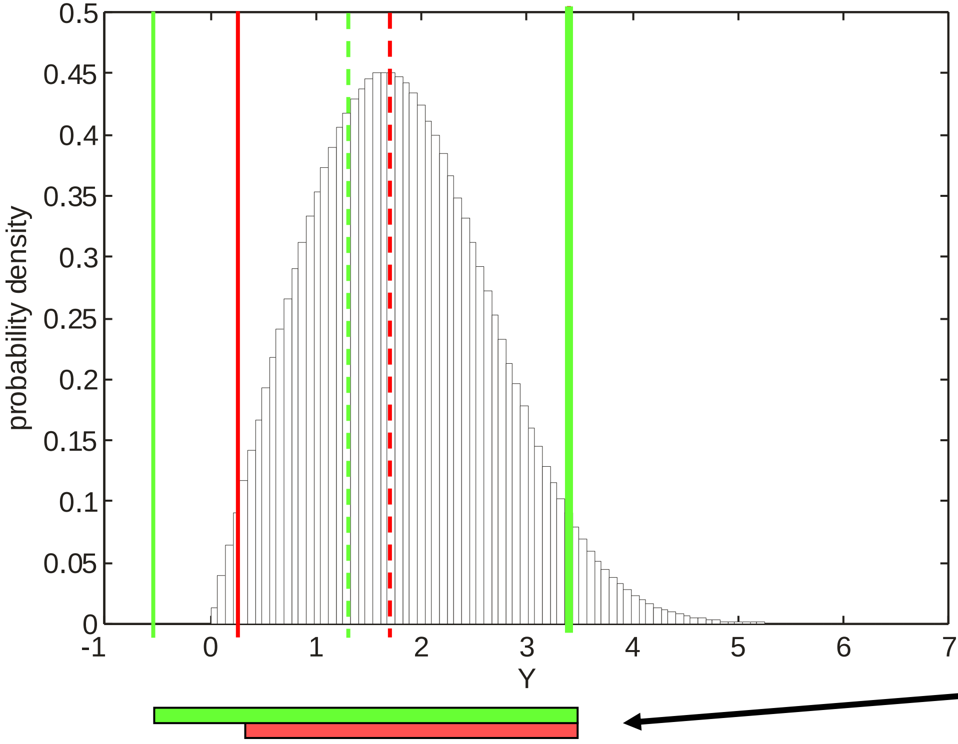
examples

model: $Y = \sqrt{X_1^2 + X_2^2}$

best estimate: $x_1 = x_2 = 1$

uncertainty: $u(x_1) = u(x_2) = 1$

$\Rightarrow$ Gaussian distribution (uncorrelated)



	y	$u(y)$
MCM	1.81	0.845
GUM	1.41	1.000

coverage interval (95%)

G. Wübbeler, M. Krystek, C. Elster (2008). Meas. Sci. Technol. 19, 084009

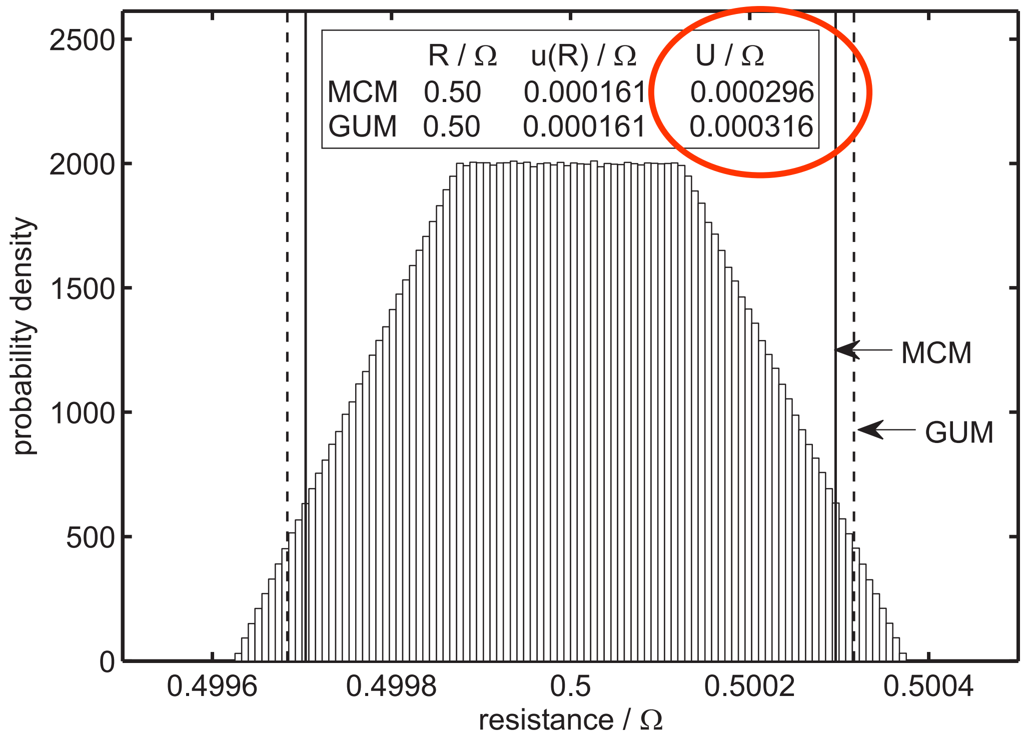
examples

model: $R = U / I$

best estimate: 1 V and 2 A

uncertainty: $\frac{0.001}{\sqrt{12}}$ V, $\frac{0.001}{\sqrt{12}}$ A

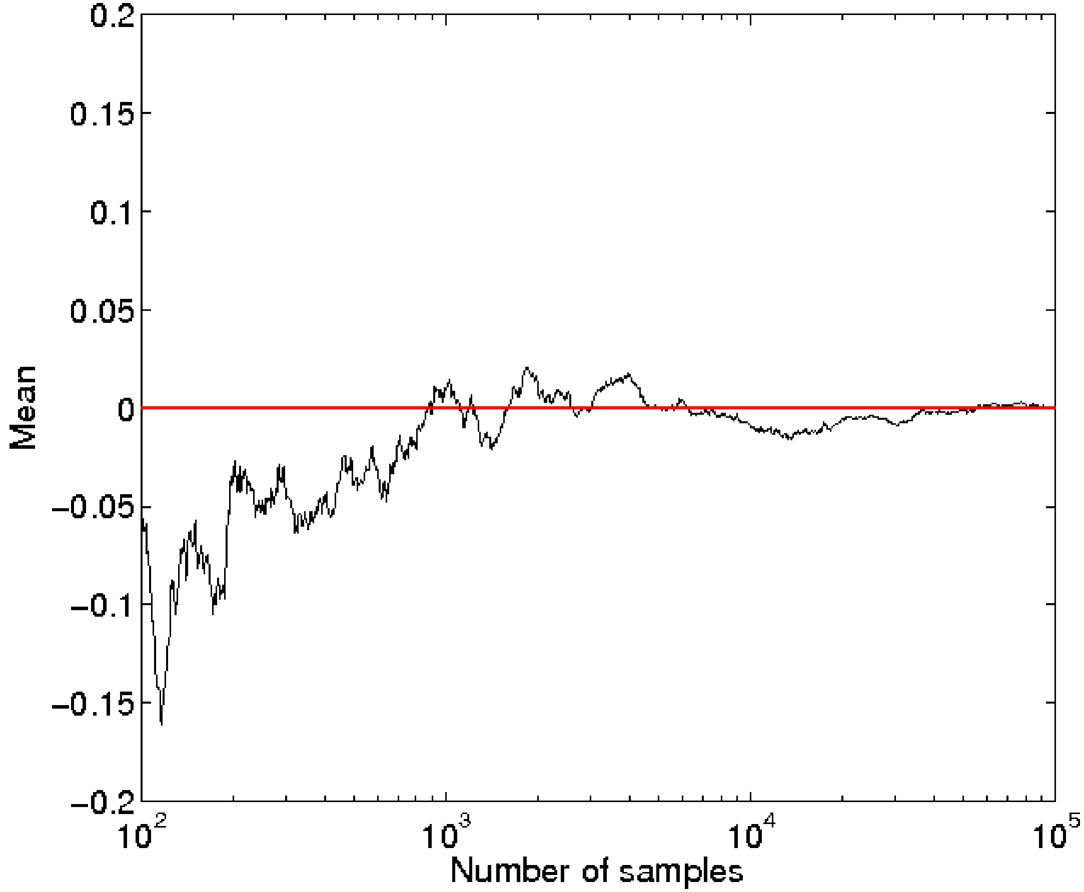
$\Rightarrow g_{x_i}(\xi_i)$ rectangular (uncorrelated)



Monte-Carlo Methode (MCM)

examples

$$X_i \sim N(0,1)$$

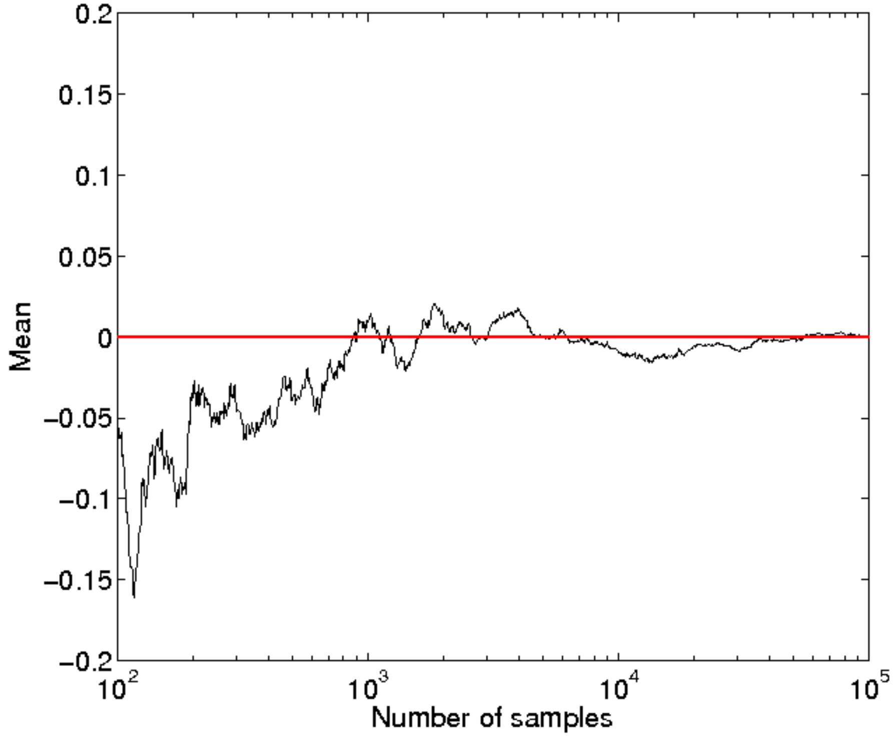


law of

large numbers

examples

$$X_i \sim N(0,1)$$



MCM results vary randomly

conclusion

- MCM is a powerful tool to determine measurement uncertainty

- In case of discrepancy between the results from GUM/GUF and GUM S1, the GUM S1 result has to be taken

- GUM/GUF is included in the GUM S1 as a special linearized case

Thank you for your attention

Questions ???

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